

Phase structure of twisted Eguchi-Kawai model

Tatsuo Azeanagi and Tomoyoshi Hirata

*Department of Physics, Kyoto University,
Kyoto 606-8502, Japan*

E-mail: aze@gauge.scphys.kyoto-u.ac.jp, hirata@gauge.scphys.kyoto-u.ac.jp

Masanori Hanada

*Theoretical Physics Laboratory, RIKEN Nishina Center,
Wako, Saitama 351-0198, Japan, and*

*Department of Particle Physics, Weizmann Institute of Science,
Rehovot 76100, Israel*

E-mail: masanori.hanada@weizmann.ac.il

Tomomi Ishikawa

*RIKEN BNL Research Center, Brookhaven National Laboratory,
Upton, New York 11973, U.S.A.*

E-mail: tomomi@quark.phy.bnl.gov

ABSTRACT: Twisted Eguchi-Kawai model is a useful tool for studying the large- N gauge theory. It can also provide a nonperturbative formulation of the gauge theory on noncommutative spaces. Recently it was found that the \mathbb{Z}_N^4 symmetry in this model, which is crucial for the above applications, can break spontaneously in the intermediate coupling region. In this article, we study the phase structure of this model using the Monte-Carlo simulation. In particular, we elaborately investigate the symmetry breaking point from the weak coupling side. The simulation results show that we cannot take a continuum limit for this model.

KEYWORDS: M(atric) Theories, Lattice Gauge Field Theories, Lattice Quantum Field Theory.

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1. Introduction

The large- N gauge theories provide fruitful features to both phenomenology and string theory. They are simplified in the large- N limit while preserving essential features of QCD [1]. Additionally, dimensional reductions of ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory (matrix model) are expected to provide nonperturbative formulations of superstring theory [2–5], and can also be regarded as effective actions of D-branes [6]. Furthermore, their twisted reduced versions, which we study in this article, can provide a nonperturbative formulation of the gauge theories on noncommutative spaces (NCYM) [7, 8]. In order to study the nonperturbative nature of these theories, numerical simulations using lattice regularizations are quite efficient. (Non-lattice simulations are also applicable for the reduced models. See references [9, 10] for the recent progress.)

In the large- N limit there is an equivalence between the gauge theory and its zero-dimensional reduction, which is known as *Eguchi-Kawai equivalence* [11]. Here, we consider the $SU(N)$ gauge theory (YM) on D -dimensional periodic lattice with the Wilson’s plaquette action

$$S_W = -\beta N \sum_x \sum_{\mu \neq \nu} \text{Tr} U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x), \quad (1.1)$$

where $U_\mu(x)$ ($\mu = 1, \dots, D$) $\in \text{SU}(N)$ are link variables and β is the inverse of the bare 't Hooft coupling. In the large- N limit the space-time degrees of freedom can be neglected, and then this theory can be equivalent to a model defined on a single hyper-cube,

$$S_{\text{EK}} = -\beta N \sum_{\mu \neq \nu} \text{Tr} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger, \tag{1.2}$$

which is called the Eguchi-Kawai model (EK model). The equality was shown by observing that the Schwinger-Dyson equations for Wilson loops (loop equations) in both theories are the same. In the EK model the loop equations can naively have open Wilson line terms, which do not exist in the original gauge theory side due to the gauge invariance. Therefore we need to assume that the global \mathbb{Z}_N^D symmetry

$$U_\mu \rightarrow e^{i\theta_\mu} U_\mu, \tag{1.3}$$

which eliminates the non-zero expectation value of the open Wilson lines, is not broken spontaneously. However, soon after the discovery of the equivalence, it was found that the \mathbb{Z}_N^D symmetry is actually broken for $D > 2$ in the weak coupling region [12]. Although the naive EK equivalence does not hold, modifications were proposed for this issue. They are quenched Eguchi-Kawai model (QEK model) [12–14] and twisted Eguchi-Kawai model (TEK model) [15]. Historically, most of the works previously done were based on the TEK model because this model is theoretically interesting and numerically more practical (and this model describes the NCYM as mentioned before).

In the TEK model, twisted boundary conditions are imposed and then the \mathbb{Z}_N^D symmetry is ensured in the weak coupling limit. It is not obvious whether the symmetry is broken or not in the intermediate coupling region. There is no guarantee for not violating the symmetry. Numerical simulations in the 1980s, however, suggested that the \mathbb{Z}_N^D symmetry is not broken throughout the whole coupling region. Then we have believed that the TEK model actually describes the large- N limit of the gauge theory.

Recently some indication about the \mathbb{Z}_N^D symmetry breaking was surprisingly reported in several context around the TEK model [16–19]. The most relevant discussion for the present article was done by Teper and Vairinhos in [17]¹. They showed that the \mathbb{Z}_N^D symmetry is really broken in the intermediate coupling region by the Monte-Carlo simulation for the $D = 4$ TEK model with the standard twist. Our work in this article is along this line and we mainly concentrate on investigating locations of the symmetry breaking from the weak coupling side in (β, N) plane. By the Monte-Carlo simulation we clarify the linear behavior of critical lattice coupling

$$\beta_c^L \sim L^2, \tag{1.4}$$

where β_c^L represents critical lattice coupling from the weak coupling side and L is the lattice size we have considered. This result means that *the continuum limit of the planar gauge*

¹In [18, 19] a similar model with two commutative and two noncommutative dimensions were studied in the context of NCYM. In this case the instability of \mathbb{Z}_N preserving vacuum was observed even in a perturbative calculation. This instability arises due to UV/IR mixing.

theory cannot be described by the TEK model from the argument of the scaling behavior around the weak coupling limit. This discussion can be also applied to the NCYM case.

This article is organized as follows. In the next section we review the TEK model briefly and fix our setup. In section 3 we show the numerical results for the \mathbb{Z}_N^D symmetry breaking of the TEK model and find the scaling behavior (1.4). In section 4 we give the validation for the numerical result, and also discuss whether the TEK model has a continuum limit or not.

2. Twisted Eguchi-Kawai model

2.1 Action and Wilson loop

In this study, we treat the $D = 4$ case. The TEK model [15] is a matrix model defined by the partition function

$$Z_{\text{TEK}} = \int \prod_{\mu=1}^4 dU_{\mu} \exp(-S_{\text{TEK}}), \quad (2.1)$$

with the action

$$S_{\text{TEK}} = -\beta N \sum_{\mu \neq \nu} Z_{\mu\nu} \text{Tr} U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}, \quad (2.2)$$

where U_{μ} and dU_{μ} ($\mu = 1, 2, 3, 4$) are link variables and Haar measure. The phase factors $Z_{\mu\nu}$ are

$$Z_{\mu\nu} = \exp(2\pi i n_{\mu\nu}/N), \quad n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z}_N. \quad (2.3)$$

The Wilson loop operator also contains the phase $Z(\mathcal{C})$ as

$$W_{\text{TEK}}(\mathcal{C}) \equiv Z(\mathcal{C}) \langle \hat{W}(\mathcal{C}) \rangle, \quad (2.4)$$

where $\hat{W}(\mathcal{C})$ is the trace of the product of link variables along a contour \mathcal{C} and $Z(\mathcal{C})$ is the product of $Z_{\mu\nu}$'s which correspond to the plaquettes in a surface whose boundary is \mathcal{C} . This model is obtained by dimensional reduction of the Wilson's lattice gauge theory with the twisted boundary condition. With these definitions, the loop equations in the TEK model take the same form as those in the ordinary lattice gauge theory if the \mathbb{Z}_N^4 symmetry, which we discuss in section 2.3, is not broken.

2.2 Twist prescriptions and classical solutions

In the weak coupling limit, the path-integral is dominated by the configuration which gives the minimum to the action. This configuration $U_{\mu}^{(0)} = \Gamma_{\mu}$ satisfies the 't Hooft algebra

$$\Gamma_{\mu} \Gamma_{\nu} = Z_{\nu\mu} \Gamma_{\nu} \Gamma_{\mu}, \quad (2.5)$$

and is called "twist-eater". The most popular twist might be the minimal symmetric twist (standard twist)

$$n_{\mu\nu} = \begin{pmatrix} 0 & L & L & L \\ -L & 0 & L & L \\ -L & -L & 0 & L \\ -L & -L & -L & 0 \end{pmatrix}, \quad N = L^2. \quad (2.6)$$

This twist represents L^4 lattice. In order to construct the classical solution for this twist, it is convenient to use the $SL(4, \mathbb{Z})$ transformation for the coordinates on \mathbb{T}^4 [20]. Using the $SL(4, \mathbb{Z})$ transformation we can always rewrite the $n_{\mu\nu}$ in the skew-diagonal form

$$n_{\mu\nu} \longrightarrow n'_{\mu\nu} = V^T n_{\mu\nu} V = \left(\begin{array}{cc|cc} 0 & L & 0 & 0 \\ -L & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & L \\ 0 & 0 & -L & 0 \end{array} \right), \quad (2.7)$$

where V is a $SL(4, \mathbb{Z})$ transformation matrix. This form makes the construction of the twist-eater easy. Here we define $L \times L$ “shift” matrix \hat{S}_L and “clock” matrix \hat{C}_L by

$$\hat{S}_L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad \hat{C}_L = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i/L} & & & \\ & & e^{2\pi i \cdot 2/L} & & \\ & & & \ddots & \\ O & & & & e^{2\pi i(L-1)/L} \end{pmatrix}, \quad (2.8)$$

which satisfy the little 't Hooft algebra

$$\hat{C}_L \hat{S}_L = e^{-2\pi i/L} \hat{S}_L \hat{C}_L. \quad (2.9)$$

Using these matrices, the twist-eater configuration for the skew-diagonal form (2.7) is easily constructed as

$$\begin{aligned} \Gamma_1 &= \hat{C}_L \otimes \mathbb{1}_L, & \Gamma_2 &= \hat{S}_L \otimes \mathbb{1}_L, \\ \Gamma_3 &= \mathbb{1}_L \otimes \hat{C}_L, & \Gamma_4 &= \mathbb{1}_L \otimes \hat{S}_L. \end{aligned} \quad (2.10)$$

From (2.7) we can also construct the twist-eater configuration for the minimal symmetric twist (2.6) as

$$\begin{aligned} \Gamma_1 &= \hat{C}_L \otimes \mathbb{1}_L, & \Gamma_2 &= \hat{S}_L \hat{C}_L \otimes \hat{C}_L, \\ \Gamma_3 &= \hat{S}_L \hat{C}_L \otimes \hat{S}_L, & \Gamma_4 &= \hat{S}_L \otimes \mathbb{1}_L. \end{aligned} \quad (2.11)$$

Although these forms are different only by the coordinate transformation, they can give different results except the weak coupling limit as seen in next section.

Another kind of the twist we consider in this article is

$$n_{\mu\nu} = \left(\begin{array}{cc|cc} 0 & mL & 0 & 0 \\ -mL & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & mL \\ 0 & 0 & -mL & 0 \end{array} \right), \quad N = mL^2 \quad (2.12)$$

with classical solution

$$\begin{aligned} \Gamma_1 &= \hat{C}_L \otimes \mathbb{1}_L \otimes \mathbb{1}_m, & \Gamma_2 &= \hat{S}_L \otimes \mathbb{1}_L \otimes \mathbb{1}_m, \\ \Gamma_3 &= \mathbb{1}_L \otimes \hat{C}_L \otimes \mathbb{1}_m, & \Gamma_4 &= \mathbb{1}_L \otimes \hat{S}_L \otimes \mathbb{1}_m. \end{aligned} \quad (2.13)$$

While we write the twist using the skew-diagonal form here, we can always rewrite it in the symmetric form by the $SL(4, \mathbb{Z})$ transformation. We call this twist “generic twist” in this article, and the minimal twists (2.6) and (2.7) are particular cases ($m = 1$) of the generic twist. As is well known, the TEK model can describe the NCYM theory [7, 8]. Expanding the matrix model around noncommutative tori background, we can obtain noncommutative $U(m)$ Yang-Mills theory on fuzzy tori. (Note that this interpretation is possible even at finite- N .) Because fuzzy torus can be used as a regularization of fuzzy \mathbb{R}^4 , it is naively possible to give a nonperturbative formulation of the NCYM on fuzzy \mathbb{R}^4 by taking a suitable large- N limit in the TEK model. (See appendix A for details.) However, we will see later it is not the case because of the \mathbb{Z}_N^4 symmetry breaking. In the NCYM interpretation the shift and clock matrices can be regarded as matrix realization of a fuzzy torus. From this point of view, twist prescription (2.12) provides YM theories on m -coincident four-dimensional fuzzy tori.

2.3 \mathbb{Z}_N^4 symmetry

The \mathbb{Z}_N^4 symmetry plays a crucial role in the Eguchi-Kawai equivalence. Generally, the YM theory with a periodic boundary condition has a critical size. If we shrink the volume of the system beyond the critical size, we encounter the center symmetry breaking, which is just the same as the finite temperature system. In the EK model, which is a single hyper-cubic model, the critical size corresponds to $\beta_c \sim 0.19$ in the lattice coupling. In the region less than the β_c — the strong coupling region — the center symmetry \mathbb{Z}_N^4 is maintained. On the other hand, in the region larger than β_c — the weak coupling region — the symmetry is spontaneously broken, and then the EK equivalence does not hold.

The TEK model avoids this problem by imposing the twisted boundary condition on the system instead of the periodic one. In the weak coupling limit the path integral is dominated by the vacuum configuration, which is twist-eater configurations, as we already mentioned. These configurations are invariant under global \mathbb{Z}_L^4 transformation

$$U_\mu \rightarrow e^{i\theta_\mu} U_\mu, \quad e^{i\theta_\mu} \in \mathbb{Z}_L, \quad (2.14)$$

which is regarded as the $U(1)^4$ symmetry in the large- N limit. As a result, $W_{\text{TEK}}(\mathcal{C})$ is zero if \mathcal{C} is an open contour in the weak coupling limit.

A key point is that the solution for this problem is obvious only at the classical level. That is to say, there is no guarantee to maintain the \mathbb{Z}_L^4 symmetry if we take into account the quantum fluctuation. Going away from the weak coupling limit, the configurations fluctuate around the twist-eater. The situation can be displayed in the eigenvalue distribution of the link variables. In the weak coupling limit the N eigenvalues distribute regularly and uniformly on the unit circle in the complex plane, and then they are \mathbb{Z}_L symmetric. If we decrease β , the eigenvalues begin to fluctuate around the location of the twist-eater. If the fluctuation is not too large, the \mathbb{Z}_L symmetric distribution is maintained. However, large fluctuation can make the uniform distribution shrink to a point, which corresponds to $U_\mu = \mathbb{1}_N$ configuration. In the strong coupling region the distribution is randomly uniform, and then the symmetry is restored.

Although there is no guarantee to maintain the \mathbb{Z}_N^4 symmetry in the intermediate coupling region, the 1980s numerical simulations suggested that the symmetry was unbroken. And this caused us to believe that the EK equivalence in the TEK model does hold throughout the whole coupling region.

2.4 Limiting procedure

As is well known, the scaling of the YM lattice theory behaves as $\beta \sim \log a^{-1}$ around the weak coupling limit, where a is the lattice spacing, and which is obtained by one-loop perturbative calculation of the renormalization group equation. If we wish to construct the TEK model which corresponds to the YM theory by the EK equivalence, the scaling of the TEK model should obey that of the YM theory. In the TEK model, the lattice size L relates to N . (For the twist we consider in this article, the relation is $N = mL^2$.) Then, the YM system with fixed physical size $l = aL$ can be obtained by the scaling

$$\beta \sim \log a^{-1} \sim \log N. \tag{2.15}$$

In order to obtain the large- N limit with infinite volume, we should increase β slower than the scaling (2.15). If it is not the case, the system shrinks to a point.

In the case of the NCYM, the scaling near the weak coupling limit is essentially same as the YM theory, that is, $\beta \sim \log a^{-1}$. (See appendix A.) But if we wish to make the TEK model corresponding to the NCYM, there is a constraint $a^2L = al = \text{fixed}$, which means that we take a scheme in which the noncommutative parameter θ is fixed. Then, both the continuum limit and the infinite volume limit are simultaneously taken (double scaling limit). Regardless of difference of the constraint, the scaling for the NCYM we should take is the same as that of the ordinary YM (2.15) by the nature of the logarithm scaling.

3. \mathbb{Z}_N^4 symmetry breaking in the TEK model

As mentioned in the previous section, the \mathbb{Z}_N^4 symmetry breaking had not been observed in the older numerical simulation. However, there are several recent reports which indicate the symmetry breaking [16, 17, 19]. In [17], the symmetry breaking in the $D = 4$ $SU(N)$ TEK model was studied in the case of the standard twist up to $N = 144 = 12^2$. The authors of [17] performed the Monte-Carlo simulation starting both from a randomized configuration (“hot start”) and from the twist-eater solution (“cold start”). In both cases the \mathbb{Z}_N^4 symmetry begins to break at $N \geq 100 = 10^2$. At $N = 144$ the symmetry breaking and restoration patterns they observed are

$$\begin{aligned} \mathbb{Z}_N^4 &\xrightarrow{\beta_c^H} \mathbb{Z}_N^3 \longrightarrow \mathbb{Z}_N^2 \longrightarrow \mathbb{Z}_N^1 \longrightarrow \mathbb{Z}_N^0 \quad (N = 144, \text{ standard, hot start}), \\ \mathbb{Z}_N^4 &\longleftarrow \mathbb{Z}_N^3 \longleftarrow \mathbb{Z}_N^2 \longleftarrow \mathbb{Z}_N^1 \longleftarrow \mathbb{Z}_N^0 \xleftarrow{\beta_c^L} \mathbb{Z}_N^4 \quad (N = 144, \text{ standard, cold start}), \end{aligned} \tag{3.1}$$

where β_c^H and β_c^L are the first breaking point for the hot start and that for cold start, respectively. Note that although there is recovery of the symmetry for the cold start, the symmetry remains broken for the hot start.

N	L	β_c^L	N	L	β_c^L
100	10	0.3525 ± 0.0025	225	15	0.4575 ± 0.0025
121	11	0.3625 ± 0.0025	256	16	0.4875 ± 0.0025
144	12	0.3775 ± 0.0025	289	17	0.5275 ± 0.0025
169	13	0.3975 ± 0.0025	324	18	0.5675 ± 0.0025
196	14	0.4225 ± 0.0025			

Table 1: Critical lattice coupling from the weak coupling side β_c^L for the minimal symmetric twist.

In this section we show the results of the numerical simulation for this symmetry breaking phenomena. In order to argue about the possibility of the continuum and large- N limiting procedure for this model, we mainly focus on the first breaking point for the cold start β_c^L , which depends on N .²

3.1 Simulation method

In our simulation we use the pseudo-heatbath algorithm. The algorithm is based on [21], and in each sweep over-relaxation is performed five times after multiplying SU(2) matrices. The number of sweeps is $O(1000)$ for each β . We scan the symmetry breaking on the resolution of $\Delta\beta = 0.005$, and then we always quote the value ± 0.0025 as the error due to the resolution. Note that the breaking points are ambiguous because the breakdown of the \mathbb{Z}_N^4 symmetry is a first-order transition. As an order parameter for detecting the \mathbb{Z}_N^4 breakdown, we measure the Polyakov lines

$$P_\mu \equiv \left\langle \left| \frac{1}{N} \text{Tr } U_\mu \right| \right\rangle. \tag{3.2}$$

3.2 Simulation results

Minimal symmetric twist. First of all we treat the minimal symmetric twist (2.6). This twist is the most standard and is also used in the paper [17]. In our study we only investigate the first \mathbb{Z}_N^4 symmetry breaking point from weak coupling limit, that is, β_c^L for this twist. (For more detailed information about the symmetry breaking phenomena, see [17].) The obtained results are in table 1 and plotted in figure 1. The symmetry breaking points and patterns ($\mathbb{Z}_N^4 \xrightarrow{\beta_c^L} \mathbb{Z}_N^3$ for $N = 100$; $\mathbb{Z}_N^4 \xrightarrow{\beta_c^L} \mathbb{Z}_N^0$ for $N > 100$) are consistent with the results in [17] up to $N = 144$. In this work we explore the simulation for larger N . From figure 1 we can find clear linear dependence of β_c^L on $N(= L^2)$ for $N \gtrsim 169$. The fitted result in linear function using $N \geq 169$ data is

$$\beta_c^L \sim 0.0011N + 0.21. \tag{3.3}$$

A theoretical argument for this linear behavior is discussed in section 4.

²Strictly speaking, the symmetry preserved in the weak coupling region is not \mathbb{Z}_N^4 but \mathbb{Z}_L^4 . However, \mathbb{Z}_L^4 is sufficient for the Eguchi-Kawai equivalence so we do not dare to distinguish them in this article.

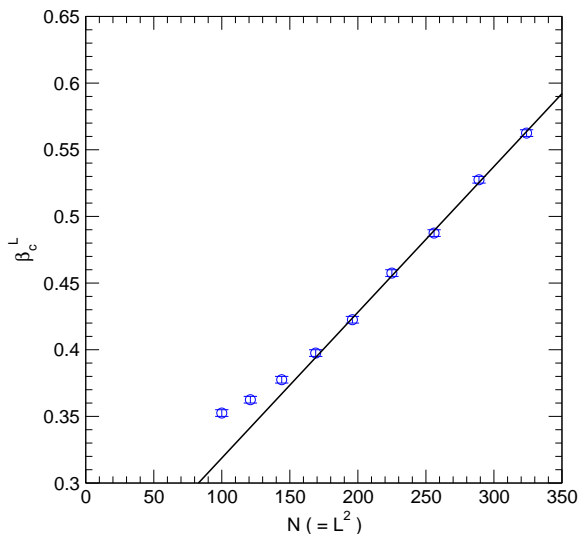


Figure 1: Plot of β_c^L versus N for the minimal symmetric twist. Fit line is equation (3.3), which is obtained using $N \geq 169$ data.

Minimal skew-diagonal twist. Twists can be always transformed into the skew-diagonal form by $SL(4, \mathbb{Z})$ transformation as we mentioned in section 2.2. As it were, the minimal symmetric twist (2.6) is equivalent to the minimal skew-diagonal twist (2.7) in the weak coupling limit. However, both forms can represent different features by taking into account the quantum fluctuation. Actually, the \mathbb{Z}_N^4 symmetry is already broken at $N = 25$. This fact enables us to observe the N -dependence of the critical points easily. Not only is the symmetry breaking point different from the symmetric form, so is the breaking and restoration pattern. Figure 2 shows the expectation value of the plaquette (top) and the Polyakov lines (besides the top) versus β for the cold start at $N = 100$. For $N \geq 100$ we find the \mathbb{Z}_N^4 symmetry breaking and restoration pattern:

$$\mathbb{Z}_N^4 \leftarrow \mathbb{Z}_N^3 \leftarrow \mathbb{Z}_N^2 \leftarrow \mathbb{Z}_N^0 \xleftarrow{\beta_c^L} \mathbb{Z}_N^4 \quad (N = 100, \text{ minimal skew-diagonal, cold start}), \quad (3.4)$$

which represents a difference from the symmetric form case (3.1). The first breaking pattern $\mathbb{Z}_N^4 \xrightarrow{\beta_c^L} \mathbb{Z}_N^0$ is, however, the same as that in the symmetric twist. (We note that for $N \leq 81$ the first breaking pattern is $\mathbb{Z}_N^4 \xrightarrow{\beta_c^L} \mathbb{Z}_N^2$, which resembles the pattern $\mathbb{Z}_N^4 \xrightarrow{\beta_c^L} \mathbb{Z}_N^3$ at $N = 100$ for the symmetric form [17].)

Table 2 shows the first breaking points for the cold start β_c^L and for the hot start β_c^H . These data are plotted in figure 3 for β_c^L and figure 4 for β_c^H . Again, we find clear linear dependence on N for β_c^L , as we found for the symmetric form. Additionally, we also find clear dependence on $1/N$ for β_c^H . The fitted results are

$$\beta_c^L \sim 0.0034N + 0.25, \quad (3.5)$$

$$\beta_c^H \sim \frac{2.9}{N} + 0.18, \quad (3.6)$$

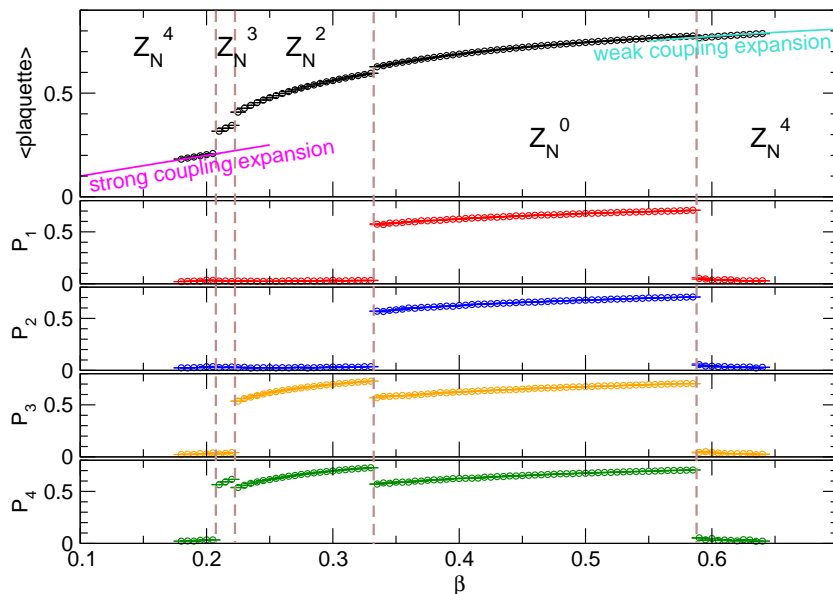


Figure 2: Expectation value of the plaquette (top) and the Polyakov line (besides the top) versus the lattice coupling β for $N = 100$ with the minimal skew-diagonal twist (cold start). As β is decreased, the \mathbb{Z}_N^4 symmetry is broken and restored as $\mathbb{Z}_N^4 \leftarrow \mathbb{Z}_N^3 \leftarrow \mathbb{Z}_N^2 \leftarrow \mathbb{Z}_N^0 \xleftarrow{\beta_c^L} \mathbb{Z}_N^4$.

N	L	β_c^H	β_c^L	N	L	β_c^H	β_c^L
9	3	-	-	64	8	0.2225 ± 0.0025	0.4625 ± 0.0025
16	4	-	-	81	9	0.2125 ± 0.0025	0.5175 ± 0.0025
25	5	0.2925 ± 0.0025	0.3625 ± 0.0025	100	10	0.2075 ± 0.0025	0.5875 ± 0.0025
36	6	0.2575 ± 0.0025	0.3925 ± 0.0025	121	11	0.2025 ± 0.0025	0.6525 ± 0.0025
49	7	0.2375 ± 0.0025	0.4225 ± 0.0025	144	12	0.1975 ± 0.0025	0.7325 ± 0.0025

Table 2: Critical lattice coupling from the weak coupling side β_c^L and from strong coupling side β_c^H for the minimal skew-diagonal twist ($m = 1$).

where we used only $N \geq 64$ data for β_c^L , whereas all data are used for β_c^H . As N is increased the β_c^H approaches a point 0.190, where the phase transition $\mathbb{Z}_N^4 \xrightarrow{\beta_c^H} \mathbb{Z}_N^3$ takes place in the original EK model. These results suggest that the quantum fluctuation is so large that the \mathbb{Z}_N^4 symmetry is broken in exactly the same region as that in the original EK model. The lines for transitions β_c^L and β_c^H seem to intersect around the bulk transition point $\beta_c^B \sim 0.35$, which corresponds to $N \sim 20$ for the twist considered here. For smaller values than $N \sim 20$, we did not observe a signal of breakdown of the \mathbb{Z}_N^4 symmetry.

Generic skew-diagonal twist. Here, we show the numerical result of the generic twist (2.12). For this twist we use the skew-diagonal form because the \mathbb{Z}_N^4 symmetry breaking occurs at smaller N than that in the symmetric form, which makes our investigation much easier.

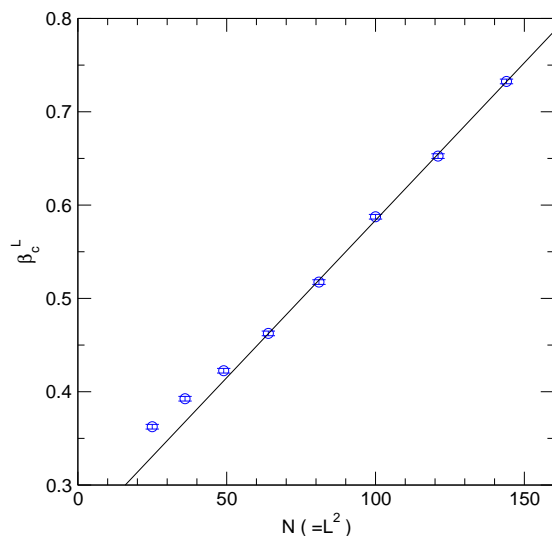


Figure 3: Plot of β_c^L versus N for the minimal skew-diagonal twist. The fit line is equation (3.5), which is obtained using $N \geq 64$ data.

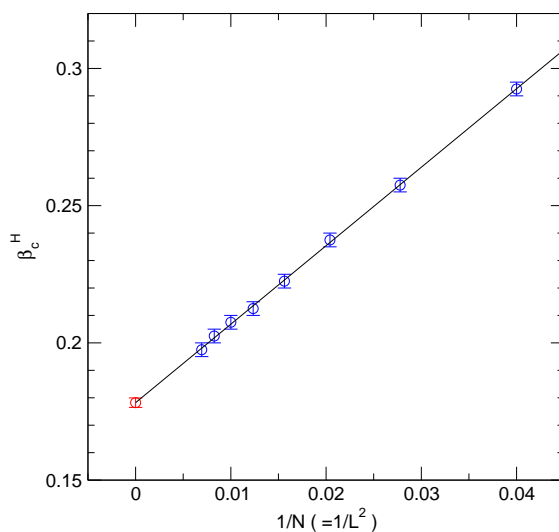


Figure 4: Plot of β_c^H versus $1/N$ for the minimal skew-diagonal twist. The fit line is equation (3.6), which is obtained using $N \geq 25$ data. Extrapolation to $1/N = 0$ gives $\beta_c^H \rightarrow 0.18$, which is close to the critical point in the original EK model, $\beta = 0.19$.

We measure β_c^L for this twist up to $m = 4$. Table 3 shows the β_c^L for $m = 2, 3, 4$ and that for $m = 1$ is presented in table 2. These data are plotted in figure 5. From this figure we can find that the β_c^L for each L are reduced as we increase m , and the dependence is linear in $1/m$. The data at $1/m = 0$ in this plot are linearly extrapolated values. An

L	$m = 2$		$m = 3$		$m = 4$	
	N	β_c^L	N	β_c^L	N	β_c^L
5	50	0.3525 ± 0.0025	75	0.3475 ± 0.0025	-	-
6	72	0.3675 ± 0.0025	108	0.3575 ± 0.0025	144	0.3525 ± 0.0025
7	98	0.3925 ± 0.0025	147	0.3875 ± 0.0025	196	0.3825 ± 0.0025
8	128	0.4375 ± 0.0025	192	0.4275 ± 0.0025	256	0.4225 ± 0.0025
9	162	0.4925 ± 0.0025	243	0.4825 ± 0.0025	324	0.4775 ± 0.0025
10	200	0.5575 ± 0.0025	300	0.5475 ± 0.0025	400	0.5425 ± 0.0025

Table 3: β_c^L for the generic skew-diagonal twist ($m = 2, 3, 4$). See also table 2 for $m = 1$.

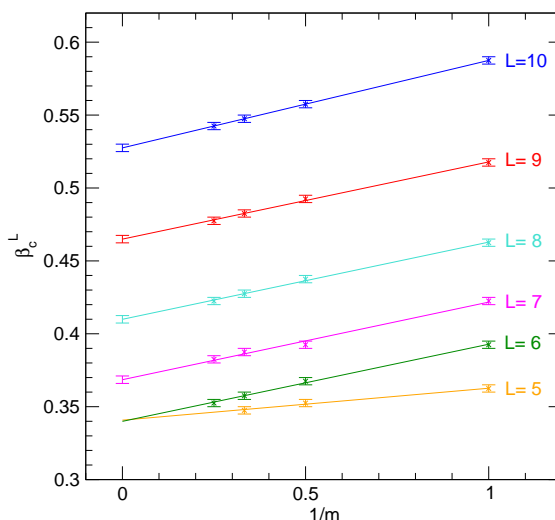


Figure 5: β_c^L versus $1/m$ for $L = 5, \dots, 10$. β_c^L for $m = \infty$ is evaluated by extrapolating these data with straight line.

interesting point is the behavior for the case $L = 5$. While the \mathbb{Z}_N^4 symmetry breaking is observed for $m = 1, 2$ and 3 , it is not seen for $m = 4$ because the β_c^L reaches the bulk transition point $\beta_c^B \sim 0.35$ by increasing m . Figure 6 represents the same data in figure 5, but the horizon axis is L^2 . As we have seen in the $m = 1$ case, the data for $L \geq 8$ are well fitted by the linear function of L^2 for each m . From these figures, we find that the data for $L \geq 8$ are well fitted globally by a function:

$$\beta_c^L \sim 0.0034L^2 + \frac{0.060}{m} + 0.19. \tag{3.7}$$

4. Discussions

In this section we discuss the numerical results obtained in the previous section and the validity of taking the large- N and continuum limit for this model.

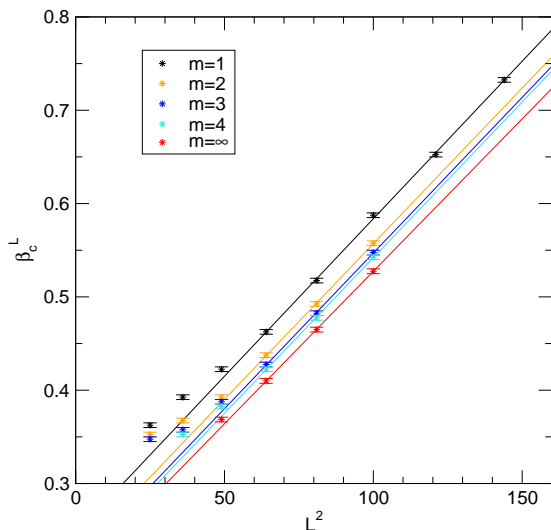


Figure 6: Scaling of β_c^L for $m = 1, 2, 3$ and 4 . We also include β_c^L for $m = \infty$, which is obtained by an extrapolation shown in figure 5.

4.1 Theoretical estimation of the \mathbb{Z}_N^4 symmetry breaking point

In the previous section we showed our numerical results. In particular, we elaborately investigated β_c^L , the first \mathbb{Z}_N^4 breaking point from the cold start. From our investigation, we found the clear linear behavior like (3.3), (3.5) and (3.7). These behaviors can be obtained through the following consideration.

Energy difference between twist-eater Γ_μ and identity $\mathbb{1}_N$ configurations. We simply assume that the \mathbb{Z}_N^4 breaking is a transition from twist-eater phase $U_\mu = \Gamma_\mu$ to identity configuration phase $U_\mu = \mathbb{1}_N$. For plainness, we consider $\mathbb{Z}_N^4 \xrightarrow{\beta_c^L} \mathbb{Z}_N^0$ type breaking here. Of course we can treat $\mathbb{Z}_N^4 \xrightarrow{\beta_c^L} \mathbb{Z}_N^3 \xrightarrow{\beta_c^L} \mathbb{Z}_N^2 \xrightarrow{\beta_c^L} \mathbb{Z}_N^1 \xrightarrow{\beta_c^L} \mathbb{Z}_N^0$ (cascade) type breaking at a β_c^L , but the obtained behavior is not different from the former type. Firstly, we focus on the classical energy difference between these configurations. The energy can be easily calculated from the action (2.2) as

$$\begin{aligned} \Delta S &= S_{\text{TEK}}[U_\mu = \mathbb{1}_N] - S_{\text{TEK}}[U_\mu = \Gamma_\mu] \\ &= \beta N^2 \sum_{\mu \neq \nu} \left\{ 1 - \cos \left(\frac{2\pi n_{\mu\nu}}{N} \right) \right\} \simeq 2\pi^2 \beta \sum_{\mu \neq \nu} n_{\mu\nu}^2. \end{aligned} \quad (4.1)$$

For the generic twist, it becomes

$$\Delta S = \begin{cases} 24\pi^2 \beta m^2 L^2 & \text{(symmetric form),} \\ 8\pi^2 \beta m^2 L^2 & \text{(skew-diagonal form).} \end{cases} \quad (4.2)$$

Note that the symmetric form is roughly three times more stable than the skew-diagonal form if both twists have equal quantum fluctuations. This is the reason that the \mathbb{Z}_N^4

symmetry breaking for the skew-diagonal form can occur at quite smaller N than that for the symmetric form, as is observed in our simulation.

Quantum fluctuations and symmetry breaking. Going away from the weak coupling limit, the system has quantum fluctuations. Here we naively expect that the \mathbb{Z}_N^4 symmetry is broken if the fluctuation around twist-eater configuration exceeds the energy difference ΔS . Because the system describes $O(N^2)$ interacting gluons, it is natural to assume that their quantum fluctuations provide $O(N^2)$ value to the effective action. For the generic twist the quantum fluctuation is $O(m^2 L^4)$. Combined with the fact (4.2), we can estimate the critical point β_c^L as

$$\beta_c^L \sim L^2, \tag{4.3}$$

which is consistent with the numerical results (3.3), (3.5) and (3.7). In addition we can explain the difference of the coefficient of L^2 in (3.3), (3.5) and (3.7) between the symmetric and the skew-diagonal form, which is roughly three times different, by the factor in (4.2).

Although the above crude estimation reproduces the linear L^2 behavior of β_c^L , we cannot explain the dependence on m . To catch the behavior completely, we need to make the discussion more sophisticated. However, we do not pursue this issue here because the m dependence can be negligible at the larger N .

This argument can be applied for other twist prescriptions like taking the twist phase as $\exp(i\pi(L+1)/L)$, which is usually used for describing noncommutative spaces. (See appendix A.)

4.2 Continuum and large- N limit

We have shown that the linear L^2 dependence of the critical point β_c^L could be explained by the theoretical discussion in this section. While our simulation is restricted in the small N region, we confirm that the behavior must continue to $N = \infty$ by combining with the discussion. Then the EK equivalence is valid only in the region $\beta > \beta_c^L \sim N$ even in the weak coupling limit and the large- N limit. As we mentioned in the section 2.4, both the ordinary YM with fixed physical volume and the NCYM theory with fixed noncommutative parameter have essentially logarithm scaling (2.15) near the weak coupling limit. Then, because β_c^L grows faster than the logarithm, the EK equivalence does not hold in the continuum limit.

5. Conclusions

In order to study the nonperturbative nature of the large- N gauge theory by lattice simulations, the large- N reduction is very useful property for saving the computational effort. In this paper, we studied the phase structure of the TEK model, which has been a major way to realize the large- N reduction. Contrary to the naive hope in old days, at least in ordinary twist prescriptions as investigated in this paper, the \mathbb{Z}_N^4 symmetry is broken even in the weak coupling region and hence a continuum limit as the planar gauge theory cannot be described by the TEK model. For the NCYM, the situation is the same. We can also consider a lot of variation for the twist prescription and the combination of reduced

and non-reduced dimension. For example, in [18, 19], four-dimensional model with two commutative and two noncommutative directions was studied using two-dimensional lattice action. However, the \mathbb{Z}_N^4 symmetry is broken also in this model, and hence we cannot take a naive continuum limit.

Another way for the reduction is the QEK model, in which the eigenvalues of the link variables are quenched. The QEK model might have no problem in principle, but its computational cost is larger than that of the TEK model. Although the TEK and QEK model are reduced models to a single hyper-cube, recent studies deviate from them. The contemporary method might be the partial reduction [22]. This work showed that the large- N reduction is valid above some critical physical size l_c . This means that for a lattice size L the reduction holds below some lattice coupling $\beta(L)$. In order to take continuum limits we should avoid the bulk transition point β_c^B , causing the condition $\beta_c^B < \beta(L)$ to be necessary. That is, there is a lower limitation for the lattice size L_c for the continuum reduction. In addition, the twist prescription is also applicable to the partial reduction [23]. Due to the twisted boundary condition, the lower limitation L_c can be reduced. Therefore, combination of the twist prescription and the partial reduction would be quite efficient in the current situation.

Note also that NCYM on fuzzy \mathbb{R}^4 could be realized by using TEK with quotient conditions [8] which give a periodic condition to eigenvalues and hence quantum fluctuation is suppressed. Further study in this direction would be important.

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A. Double scaling limit as the noncommutative Yang-Mills theory

The TEK model can be used to formulate gauge theories on noncommutative spaces non-perturbatively [7, 8, 24]. In this appendix, we give a review for the construction of the NCYM from the TEK model [7], a discussion for the scaling and some supplemental comments for our analysis.

By taking $U_\mu = e^{iaA_\mu}$, where a corresponds to the lattice spacing, and expanding the action of the TEK model (2.2), we have its continuum version

$$S_{TEK,continuum} = -\frac{1}{4g^2} \sum_{\mu \neq \nu} \text{Tr} ([A_\mu, A_\nu] - i\theta_{\mu\nu})^2 \tag{A.1}$$

up to higher order terms in a , where

$$\theta_{\mu\nu} = \frac{2\pi n_{\mu\nu}}{Na^2}, \quad \frac{1}{2g^2} = a^4 \beta N. \tag{A.2}$$

Then, by expanding the action around a classical solution of (A.1)

$$A_\mu^{(0)} = \hat{p}_\mu, \quad [\hat{p}_\mu, \hat{p}_\nu] = i\theta_{\mu\nu}, \quad (\text{A.3})$$

we obtain the U(1) NCYM on fuzzy \mathbb{R}^4 as follows. Let us define the “noncommutative coordinate” $\hat{x}^\mu = (\theta^{-1})^{\mu\nu} \hat{p}_\nu$. Then we have

$$[\hat{x}^\mu, \hat{x}^\nu] = -i(\theta^{-1})^{\mu\nu}. \quad (\text{A.4})$$

This commutation relation is the same as that of coordinate on fuzzy \mathbb{R}^4 with noncommutativity parameter θ , and hence functions of \hat{x} can be mapped to functions on fuzzy \mathbb{R}^4 . More precisely, we have the following mapping rule:

$$\begin{aligned} f(\hat{x}) = \sum_k \tilde{f}(k) e^{ik\hat{x}} &\leftrightarrow f(x) = \sum_k \tilde{f}(k) e^{ikx}, \\ f(\hat{x})g(\hat{x}) &\leftrightarrow f(x) \star g(x), \\ i[\hat{p}_\mu, \cdot] &\leftrightarrow \partial_\mu, \\ \text{Tr} &\leftrightarrow \frac{\sqrt{\det \theta}}{4\pi^2} \int d^4x, \end{aligned} \quad (\text{A.5})$$

where \star represents the noncommutative star product,

$$f(x) \star g(x) = f(x) \exp\left(-\frac{i}{2} \overleftarrow{\partial}_\mu (\theta^{-1})^{\mu\nu} \overrightarrow{\partial}_\nu\right) g(x), \quad (\text{A.6})$$

and we obtain U(1) NCYM action

$$S_{U(1)NC} = -\frac{1}{4g_{\text{NC}}^2} \int d^4x F_{\mu\nu} \star F_{\mu\nu} \quad (\text{A.7})$$

with coupling constant

$$g_{\text{NC}}^2 = 4\pi^2 g^2 / \sqrt{\det \theta}. \quad (\text{A.8})$$

In the same way, by expanding the action (A.1) around $A_\mu^{(0)} = \hat{p}_\mu \otimes \mathbb{1}_m$, U(m) NCYM can be obtained. From (2.13), it is apparent that the generic twist gives the U(m) NCYM. Intuitively, the vacuum configuration (2.13) describes m -coincident fuzzy tori and fuzzy \mathbb{R}^4 is realized as a tangent space.

In order to keep the noncommutative scale θ finite, we should take the double scaling limit with

$$a^{-1} \sim \sqrt{L} \sim N^{1/4}. \quad (\text{A.9})$$

One-loop beta function for U(m) NCYM is given by [26]³

$$\beta_{1\text{-loop}}(g_{\text{NC}}) = -\frac{2}{(4\pi)^2} \frac{11}{3} m g_{\text{NC}}^3 + O(g_{\text{NC}}^5). \quad (\text{A.10})$$

Therefore, the 't Hooft coupling β scales as

$$\beta \sim \frac{1}{g_{\text{NC}}^2} \sim \log N. \quad (\text{A.11})$$

³Renormalizability of the NCYM is a delicate problem. For example, see [25], in which the renormalizability is discussed by a perturbation expansion.

Then, the scaling we should take for the NCYM is just the same as that for the ordinary YM, and \mathbb{Z}_N^D symmetry is broken in the scaling limit. Therefore, fuzzy torus crunches to a point and hence the fuzzy \mathbb{R}^4 cannot be realized.⁴

Of course, we can also use other twist prescriptions. In order to make the periodicity of the discretized fuzzy torus correct, we usually take the twist as $\exp(i\pi(L+1)/L)$ [8]. Regardless of the difference of the twist, the conclusion might not be altered. Here we repeat the discussion in section 4.1. In this case, the \mathbb{Z}_N^4 is likely to break down to \mathbb{Z}_2^4 . The difference between potentials in twist-eater and \mathbb{Z}_2^4 -preserving configurations is

$$\Delta S \sim \beta N^2 \left\{ 1 - \cos\left(\frac{\pi}{L}\right) \right\} \sim \beta m^2 L^2, \quad (\text{A.12})$$

which is the same order as (4.1). Then the behavior of the critical point β_c^L (4.3) is not changed.

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⁴Although the fuzzy \mathbb{R}^4 cannot be realized, another kind of noncommutative space with finite physical volume may exist. In the case of a four-dimensional model with two commutative and two noncommutative directions, such a limit was found numerically [19].

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